

On the cross-section peaks for a heavy quark bound state

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Abstract

We discuss the values of resonance peaks of the cross-section of a heavy quark bound state obtained by means of a Green function method applied to a Coulombic model and compare the result to the Υ and J/ψ data.

1 Introduction

The total cross-section of a heavy quark bound state resonance in an e^+e^- annihilation is described by the Breit-Wigner formula

$$\sigma(\sqrt{s}) = \frac{3\pi}{M^2} \frac{\Gamma_{ee}\Gamma}{(\sqrt{s} - M)^2 + \Gamma^2/4} \quad (1)$$

where M is the mass of the resonance, \sqrt{s} the centre of mass energy, Γ the total width and Γ_{ee} the decay width into electrons.

For a bound state however a better analytical description of its total cross-section is given by the imaginary part of the Green function of the bound state itself [1, 2, 3]. The basic idea is to consider the Schrödinger equation of the bound state and compute its Green function

$$(\mathbf{H} - E)\mathcal{G}(\mathbf{x}, \mathbf{y}, E) = \delta(\mathbf{x} - \mathbf{y}) \quad (2)$$

where \mathbf{H} is the Hamiltonian of the system

$$\mathbf{H} = -\frac{\nabla_{\mathbf{x}}^2}{2m} + V(\mathbf{x}). \quad (3)$$

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The imaginary part of the derivative of the Green function given by (2) taken at the origin is proportional to the cross-section at threshold. The finite width of the state is taken into account by the substitution $E \rightarrow E + i\Gamma$, E being the energy offset from $2m$ threshold. So for the total cross-section we obtain

$$\sigma(E, \Gamma) \sim \sigma_{e^+e^-} \Im \mathcal{G}(0, 0, E + i\Gamma) \quad (4)$$

where $\sigma_{e^+e^-}$ accounts for the e^+e^- part of the process and depends on the energy solely by the function $1/s$ if the energy of the process is below the Z threshold. That is, this formula involves only the photon propagator and not the Z exchange together with its interference effect. The other factors depend on the particular process taken into account and are not universal [4, 5]. We have therefore factored eq. (4) in a product of terms, the first one for the e^+e^- process and the second one for the hadronic part of cross-section.

2 The Green function method

The next step is to compute the Green function for a realistic model. The Coulombic potential

$$V(r) = -\frac{4}{3} \frac{\alpha_s}{r} \quad (5)$$

where $r = |\mathbf{x}|$ provides an integrable system which is quite realistic for a heavy quark bound state provided the QCD energy scale of α_s is set to the inverse of Born radius, $r_B = 2/(3m\alpha_s^2)$, see [4] and references therein. In this case we are able to provide a fully analytical solution for the bound state energy levels, namely

$$E_n = -\frac{4}{9} \frac{m\alpha_s^2}{n^2} \quad (6)$$

The solution to problem (2) in the case of S wave is given in [5] with a slight notation change

$$\begin{aligned} \mathcal{G}(0, 0, E + i\Gamma) &= \frac{m}{4\pi} \left[-2\lambda \left(\left(\frac{k}{2\lambda} \right) + \right. \right. \\ &\quad \left. \left. + \log \left(\frac{k}{\mu} \right) + \psi(1 - \nu) + 2\gamma - 1 \right) \right] \end{aligned} \quad (7)$$

where $k = -\sqrt{m(E + i\Gamma)}$, $\lambda = 2\alpha_s m/3$ and the wavenumber $\nu = \lambda/k$, $E = \sqrt{s} - 2m$. The ψ is the logarithmic derivative of Euler's Gamma

function $\Gamma(x)$, $\gamma \simeq 0.57721$ is Euler's constant and μ is a soft scale that cancels out in the determination of physical observables.

By inspection of eq. (7) we see that the main contribution for energies E close to the one of bound states (6) is given by the ψ function, which has simple poles for negative integers $-n$ in the complex plane:

$$\begin{aligned} \psi(z) &= -\frac{1}{z+n} + \psi(n+1) + \\ &\sum_{k=1}^{\infty} \left[\frac{\psi^{(k)}(1)}{k!} + \zeta(k+1) - \zeta(k+1, n+1) \right] (z+n)^k \end{aligned} \quad (8)$$

for $z \rightarrow -n$, where $\psi^{(k)}(z)$ is the k -th derivative of the $\psi(z)$ function with respect to z , the so-called polygamma function; $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$ is the Riemann zeta function, while $\zeta(z, q) = \sum_{k=1}^{\infty} (k+q)^{-z}$ is the Hurwitz zeta function [6].

3 Results of the Coulombic model for Υ state

As previously discussed in sec. 1 the cross-section of a heavy quark resonance below Z mass could be written as

$$\sigma(E, \Gamma) \sim \frac{1}{s} \Im \mathcal{G}(0, 0, E + i\Gamma) \quad (9)$$

where in our particular model of Coulombic interaction \mathcal{G} is given explicitly by formula (7).

With (9) we are going to compute the ratio of the $2S$ and $1S$ resonance peaks respectively, which should be independent upon the particular bound state chosen for this evaluation. In fact one obtains the following expression

$$\frac{\sigma(E_{2S}, \Gamma_{2S})}{\sigma(E_{1S}, \Gamma_{1S})} = \frac{M_{1S}^2}{M_{2S}^2} \times \frac{\Im \mathcal{G}(0, 0, E_{2S} + i\Gamma_{2S})}{\Im \mathcal{G}(0, 0, E_{1S} + i\Gamma_{1S})} \quad (10)$$

which will be renamed as $\sigma(2)/\sigma(1)$ for sake of brevity. The first term of RHS of eq. (10) is close to 1, and from eq. (6) the mass of the bound state is given by

$$M_n = 2m + E_n = 2m \left(1 - \frac{2\alpha_s^2}{9n^2} \right) \quad (11)$$

so that the ratio M_{2S}^2/M_{1S}^2 is given by $1 + \alpha_s^2/3 + \mathcal{O}(\alpha_s^4)$.

Defining a suitable variable depending on a generic width value $t = \Gamma/E_1$, with E_1 given by eq. (6) for $n = 1$, we obtain the fairly elegant expression

$$\begin{aligned} \frac{\sigma(2)}{\sigma(1)} &= \frac{1}{8} + \frac{(42\pi^2 + 425)t^2}{128} - \\ &- \frac{(147384\zeta(3) + 1134\pi^4 + 11096\pi^2 + 36545)t^4}{3072} + \\ &+ \mathcal{O}(t^6) \end{aligned} \quad (12)$$

which would suggest a value close to 1/8 for this ratio. Unfortunately the (12) is a very slowly convergent series as its numeric evaluation shows:

$$\frac{\sigma(2)}{\sigma(1)} = 0.125 + 6.559t^2 - 48.877t^4 + \mathcal{O}(t^6) \quad (13)$$

that is, the coefficient of $\mathcal{O}(t^k)$ term grows more than $k!$, suggesting to evaluate the result with the full expression of (7) for the Green function.

We will make use now of this method with Υ resonances data. This $b\bar{b}$ bound state is the ideal candidate for this method: it is rather heavy, $m_b \sim 5$ GeV, so that it should allow us to neglect the confining linear term proportional to r in the QCD potential [7, 8, 9].

For the $\Upsilon(1S)$ PDG data [10] give us:

$$\begin{aligned} M_{\Upsilon(1S)} &= 9.46030 \pm 0.00026 \text{ GeV} , \\ \Gamma(1S) &= 54.02 \pm 1.25 \text{ keV} , \\ \Gamma_{ee}(1S) &= 1.340 \pm 0.018 \text{ keV} ; \end{aligned} \quad (14)$$

while for the $\Upsilon(2S)$ one reads

$$\begin{aligned} M_{\Upsilon(2S)} &= 10.02326 \pm 0.00031 \text{ GeV} , \\ \Gamma(2S) &= 31.98 \pm 2.63 \text{ keV} , \\ \Gamma_{ee}(2S) &= 0.612 \pm 0.011 \text{ keV} . \end{aligned} \quad (15)$$

Plugging all those data into the Coulombic model (5) and eq. (10) we compute the ratio of the first two peaks for the Υ resonance obtaining the value

$$\frac{\sigma(2)}{\sigma(1)} = 0.211 \pm 0.024 . \quad (16)$$

This result is very sensitive to the exact total width value, and depends much less on both the mass of the state and the exact α_s coupling of the Coulombic model. It differs from first term of the series expansion (12) by approximately a factor of 2 proving its slow convergence despite the smallness of its parameter t , of the order of 10^{-4} .

4 Comparison with data

We are ready now to compare our results with the Υ measurements. From eq. (1) the maximal value of the Breit–Wigner resonance cross-section is given by the expression

$$\sigma_{max} = \frac{12\pi}{M^2} \frac{\Gamma_{ee}}{\Gamma} . \quad (17)$$

Using again the experimental data of eqs. (14), (15) we have the following value for the measured ratio:

$$\left(\frac{\sigma(2)}{\sigma(1)} \right)_{exp} = 0.685 \pm 0.094 , \quad (18)$$

which is quite sensitive to the exact determination of the widths. It is clear that this value doesn't agree with (16); some *caveats* are in order here.

According to [10, 11], because of ISR (initial state radiation) and Beamstrahlung effect the observed line shape is not simply given by eq. (1) but a convolution of the Υ width of $\mathcal{O}(\text{keV})$, of ISR and of beam energy spread of $\mathcal{O}(\text{MeV})$. The Γ_{ee} value cannot therefore be directly measured, but is calculated from the production cross-section of Υ integrated over the incoming e^+e^- energies, that is

$$\int \sigma(e^+e^- \rightarrow \Upsilon) dE \quad (19)$$

The integral itself however is again calculated from the Breit–Wigner formula (1) in a bootstrap fashion, thus leading to a heavily model dependent result of the cross-section shape.

The Coulombic model works well even at lower than Υ scales [12], there could be however some relativistic non negligible effect in this case (recall from sec. 1 that this whole method is non-relativistic). From the virial theorem applied to a Coulombic model we obtain the relation between the kinetic and potential energies average $-\langle V \rangle/2 = \langle T \rangle$ that leads to a speed estimate of the component quark inside the meson

$$\langle v^2 \rangle = \frac{8}{9} \alpha_s^2 \quad (20)$$

This brings a γ relativistic correction of about 8% at Υ scale that could change the computed ratio.

5 Further estimates: the J/ψ case

To shed some more light on this discrepancy we have described it could be useful to compare our results to the one obtained from J/ψ data.

This $\bar{c}c$ bound state is less ideal than the former Υ for our purpose, as it is lighter ($m_c \sim 1.5$ GeV), thus linear confining terms for the potential as well as relativistic corrections could have larger effects. From PDG data for J/ψ [10] we read:

$$\begin{aligned} M_{J/\psi}(1S) &= 3096.916 \pm 0.011 \text{ MeV} , \\ \Gamma(1S) &= 93.2 \pm 2.1 \text{ keV} , \\ \Gamma_{ee}(1S) &= 5.55 \pm 0.14 \text{ keV} ; \end{aligned} \quad (21)$$

while for the $\psi(2S)$ one reads

$$\begin{aligned} M_{\psi(2S)}(2S) &= 3686.09 \pm 0.04 \text{ MeV} , \\ \Gamma(2S) &= 317 \pm 9 \text{ keV} , \\ \Gamma_{ee}(2S) &= 2.38 \pm 0.04 \text{ keV} . \end{aligned} \quad (22)$$

Proceeding like described in sec. 4 we obtain for the J/ψ peak measured values

$$\left(\frac{\sigma(2)}{\sigma(1)} \right)_{exp} = 0.0890 \pm 0.0083 . \quad (23)$$

The Green function approach method detailed in sections 2 and 3 gives us

$$\frac{\sigma(2)}{\sigma(1)} = 0.0368 \pm 0.0019 . \quad (24)$$

A comparison of eq. (24) with eq. (23) shows that the result for the J/ψ case is slightly worse than the one for Υ as expected. In fact for the former case the two central values differ for more than $27\sigma_G$, while in the latter the difference is less pronounced, above $19\sigma_G$ (here σ_G refers to the error given by the Green function procedure). An estimate of the relativistic γ correction for J/ψ done in the same fashion of eq. (20) gives us a result of about 38%.

This γ correction together with the absence of a linear term in the potential could account for the larger difference seen at a lower energy scale.

We must stress again that this method of the Green function for a Coulombic potential of a Schrödinger equation furnishes us with an *exact* analytical solution given by eq. (7) only in the case of this particular interaction, and only for a non-relativistic system. The addition of further correction terms to the potential of eq. (5) like a linear confining term r or a relativistic correction of order v^2 would spoil the integrability of this problem and thus the possibility of full control over solutions. For instance, a Hamiltonian with a funnel potential like $V(r) = -4\alpha_s/(3r) + ar$ or a Hamiltonian with relativistic correction $H = p^2/(2m) - p^4/(8m^3) + V(r)$ are not exactly solvable,

and even a perturbative approach is unfit in our case as corrections to the original H of eqs. (3) and (5) are not small. Those systems would call for a purely numerical search of results which is beyond the scope of this paper.

6 Conclusions

We have compared the ratio of the first two cross-section resonance Υ peaks computed from a QCD model and the one obtained from the experiments, given respectively by 0.211 ± 0.024 and 0.685 ± 0.094 . Another comparison is done for the J/ψ case, giving a calculated value of 0.0890 ± 0.0083 and an experimental one of 0.0368 ± 0.0019 for peaks ratio. Albeit the two results do not agree with each other, it is necessary to consider that the theoretical model could need some relativistic corrections. On the other hand, the experiments do not give a direct measurement of the peaks, but rather depend on the model used to evaluate the cross-section.

Therefore it should be possible to compare again the two results using more refined methods from the theoretical model side and from the measure technique as well.

Acknowledgments

The work of N. F. was supported by the FONDAZIONE CASSA DI RISPARMIO DI SPOLETO. The author wishes to thank Y. Srivastava for constructive criticism.

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